Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN241-194-4

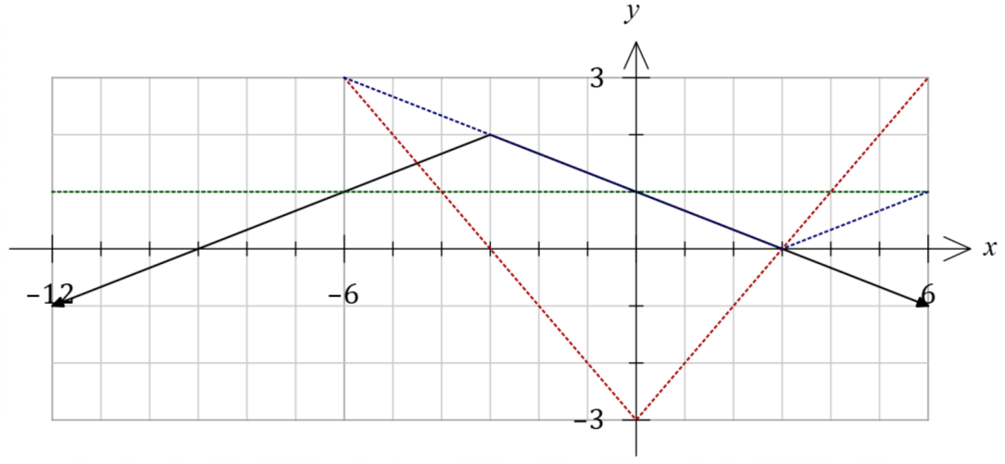
Section Two: Calculator-assumed 65% (90 Marks)

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(8 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = a - |bx + c|$ and a, b and c are all positive constants.



(a) Determine the value of each of the constants a, b and c. (3 marks)

Solution
$a = 2, \quad b = \frac{1}{3}, \quad c = 1$
Specific behaviours
✓ value of a, ✓ value of b, ✓ value of c

(b) Using the graph, or otherwise, solve

(i) $f(x) = 1$. (1 mark)

Solution
$y = 1$ intersects $f(x)$ when $x = -6, x = 0$.
Specific behaviours
✓ correct solution

(ii) $f(x) = |x| - 3$. (2 marks)

Solution
$y = |x| - 3$ intersects $f(x)$ when $x = -4.5, x = 3$.
Specific behaviours
✓ indicates $y = |x| - 3$ on graph ✓ correct solution

(iii) $3f(x) = |x - 3|$. (2 marks)

Solution
$y = \frac{1}{3}|x - 3|$ intersects $f(x)$ when $-3 \le x \le 3$.
Specific behaviours
✓ indicates $y = \frac{1}{3}|x - 3|$ on graph ✓ correct range of solutions

Question 9 **(7 marks)**

Point C lies on a sphere with centre O, radius r and diameter AB.

(a) Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Use a vector method to prove that AC is perpendicular to BC. (4 marks)

Solution
$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}, \qquad \overrightarrow{BC} = \mathbf{a} + \mathbf{c}$ $\overrightarrow{AC} \cdot \overrightarrow{BC} = (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{a} + \mathbf{c}) = \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{c} = \|\mathbf{c}\|^2 - \|\mathbf{a}\|^2$ But A, C points on sphere so that $\|\mathbf{a}\| = \|\mathbf{c}\| = r$.   Hence $\overrightarrow{AC} \cdot \overrightarrow{BC} = \|\mathbf{c}\|^2 - \|\mathbf{a}\|^2 = r^2 - r^2 = 0$ and since $\|\overrightarrow{AC}\| \neq 0$ and $\|\overrightarrow{BC}\| \neq 0$ we deduce that the angle between AC and BC must be $90°$.
Specific behaviours
✓ correct vectors for $\overrightarrow{AC}$ and $\overrightarrow{BC}$ ✓ forms and expands scalar product   ✓ uses $\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2$ explains that $\|\mathbf{a}\| = \|\mathbf{c}\| = r$ ✓ deduces perpendicularity

(b) If the position vectors of A, B and C are $\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} k \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$ respectively, determine the value of the constant k. (3 marks)

Solution
$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}, \quad \overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} k \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-k \\ 5 \\ -4 \end{pmatrix}$ $\overrightarrow{AC} \cdot \overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 2-k \\ 5 \\ -4 \end{pmatrix} = 2k + 40$ Hence $2k + 40 = 0 \Rightarrow k = -20$.
Specific behaviours
✓ vectors for $\overrightarrow{AC}$ and $\overrightarrow{BC}$ ✓ calculates scalar product   ✓ correct value of k

Question 10 **(7 marks)**

(a) Solve the equation $81z^4 + i = 0$, giving exact solutions in the form $r\,\text{cis}\,\theta, -\pi < \theta \leq \pi$. (4 marks)

Solution
$z^4 = -\frac{1}{81}i = \frac{1}{81}\text{cis}\left(-\frac{\pi}{2}\right)$ $z = \left(\frac{1}{81}\right)^{\frac{1}{4}}\text{cis}\left(-\frac{\pi + 4k\pi}{2 \times 4}\right), \quad k \in \mathbb{Z}$ $z = \frac{1}{3}\text{cis}\left(-\frac{5\pi}{8}\right), \quad z = \frac{1}{3}\text{cis}\left(-\frac{\pi}{8}\right), \quad z = \frac{1}{3}\text{cis}\left(\frac{3\pi}{8}\right), \quad z = \frac{1}{3}\text{cis}\left(\frac{7\pi}{8}\right)$
Specific behaviours
✓ writes in polar form $z^4 = \cdots$ with correct modulus   ✓ determines correct argument   ✓ states one correct solution   ✓ states all correct solutions

(b) One solution of the equation $z^n = 1$, where n is a positive integer, is $z = \text{cis}\left({}^{9\pi}/_{17}\right)$. If N solutions of the equation satisfy $0 < \arg(z) < {}^{\pi}/_{4}$, determine, with reasoning, the least value of N. (3 marks)

Solution
Solutions to the equation must be of the form $z = \text{cis}\left(\frac{2k\pi}{n}\right), k \in \mathbb{Z}$. Noting that before simplification the multiple of π will always be even, then the given solution can be written as $\text{cis}\left(\frac{2\times 9\pi}{34}\right)$ and hence minimum value of $n = 34$.   With this value of n and $1 \leq k \leq 4$, then $0 < \arg(z) < {}^{\pi}/_{4}$ and so the least value of $N = 4$.
Specific behaviours
✓ indicates general solution for n^{th} roots of unity   ✓ deduces value of n ✓ states correct number of solutions with required argument

Question 11 **(7 marks)**

A small body is moving with constant velocity in space so that initially it is located at $(5, -7, -5)$ and four seconds later it is at $(13, -11, 7)$, where all dimensions are in metres.

(a) Determine a vector equation for the position of the small body at time t seconds. (2 marks)

Solution
$\frac{1}{4}\left[\begin{pmatrix}13\\-11\\7\end{pmatrix}-\begin{pmatrix}5\\-7\\-5\end{pmatrix}\right]=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ $\mathbf{r}=\begin{pmatrix}5\\-7\\-5\end{pmatrix}+t\begin{pmatrix}2\\-1\\3\end{pmatrix}$
Specific behaviours
✓ calculates velocity vector   ✓ correct equation for position of body

A laser beam shines along the line with equation $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z-16}{2}$

(b) Write the vector equation of this line in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. (1 mark)

Solution
$\mathbf{r}=\begin{pmatrix}-1\\2\\16\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$
Specific behaviours
✓ correct vector form

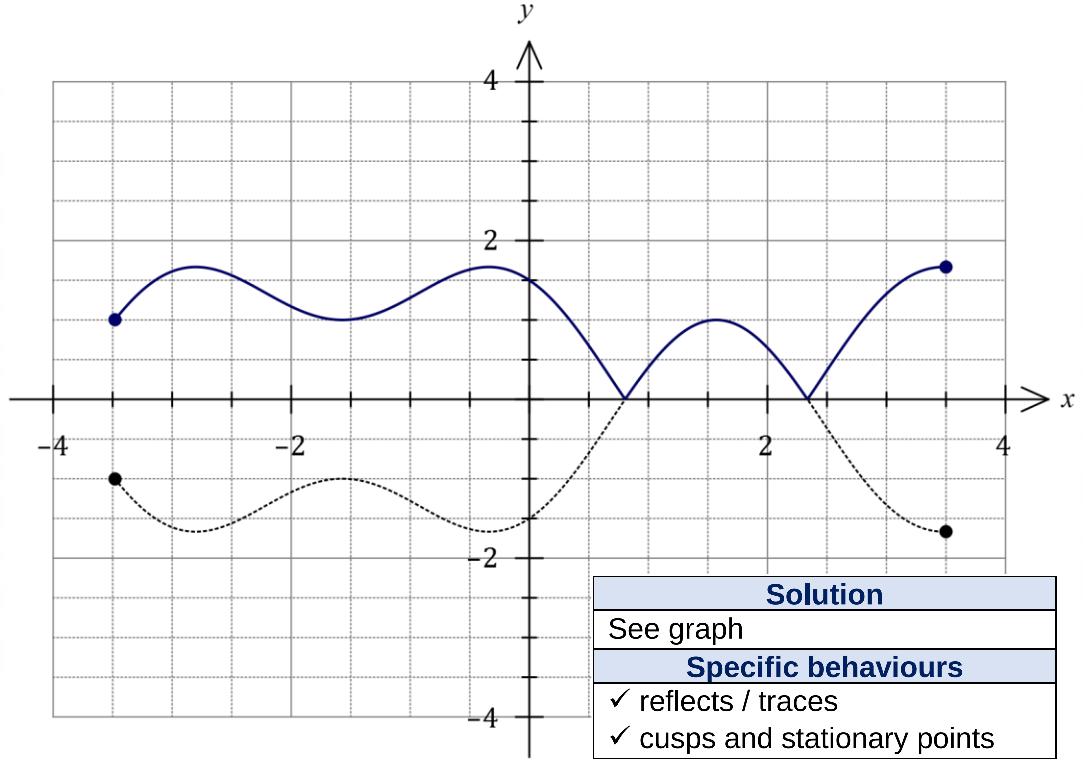
(c) Show that the small body passes through the laser beam and state where this occurs. (4 marks)

Solution
Equating $\mathbf{i}$ and $\mathbf{j}$ coefficients: $5+2t=3\lambda-1$ $-7-t=2-2\lambda$ Solving simultaneously gives $t=15, \lambda=12$.   Using these values, the body is at $\begin{pmatrix}5\\-7\\-5\end{pmatrix}+15\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}35\\-22\\40\end{pmatrix}$ and the laser passes through $\begin{pmatrix}-1\\2\\16\end{pmatrix}+12\begin{pmatrix}3\\-2\\2\end{pmatrix}=\begin{pmatrix}35\\-22\\40\end{pmatrix}$.   Hence as these points are coincident, the small body passes through the laser beam at this point.
Specific behaviours
✓ equates two coefficients   ✓ solves simultaneously   ✓ calculates both $\mathbf{k}$ coefficients or points and states they are same   ✓ states point of coincidence

Question 12 **(8 marks)**

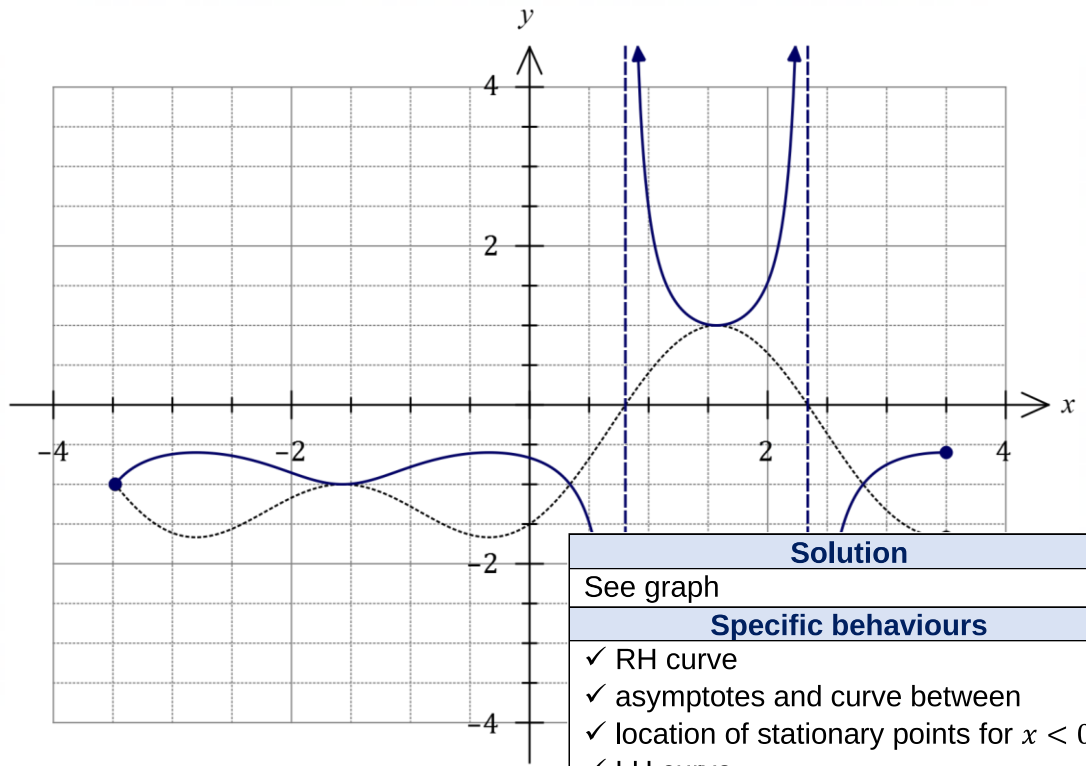
In each part of this question, the dotted curve shown is the graph of $y = f(x)$.

(a) Sketch the graph of $y = |f(x)|$. (2 marks)



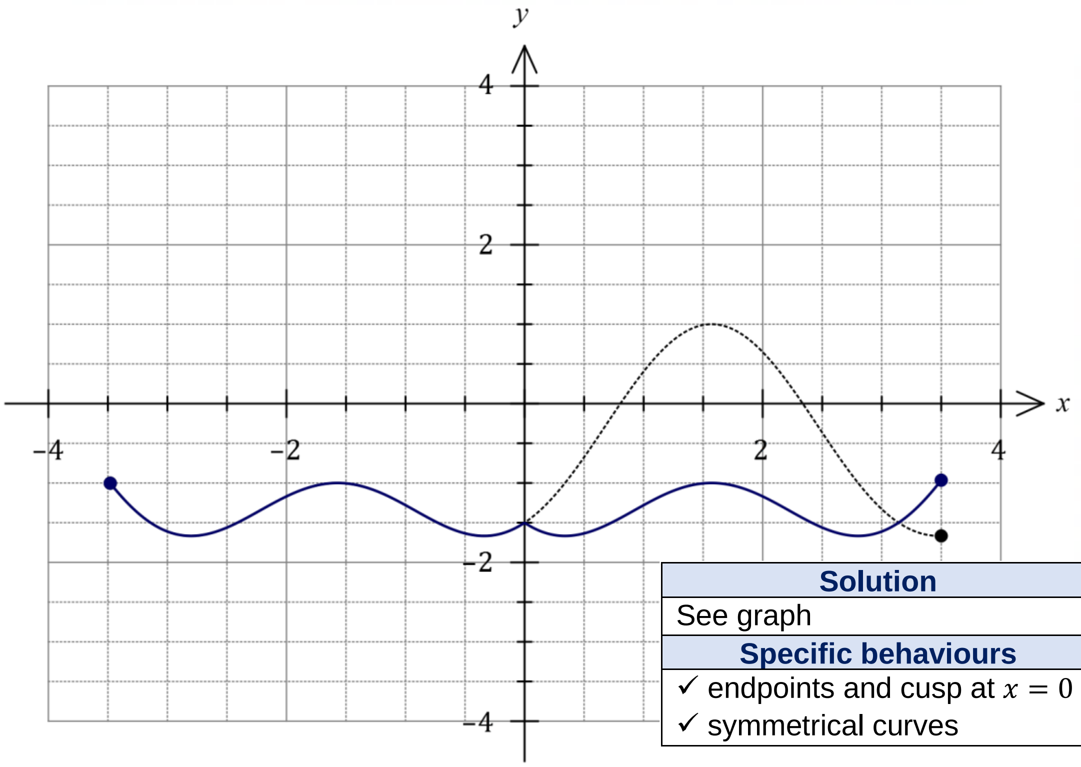
Solution
See graph
Specific behaviours
✓ reflects / traces
✓ cusps and stationary points

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$. (4 marks)



Solution
See graph
Specific behaviours
✓ RH curve
✓ asymptotes and curve between
✓ location of stationary points for $x < 0$
✓ LH curve

(c) Sketch the graph of $y = f(-|x|)$. (2 marks)

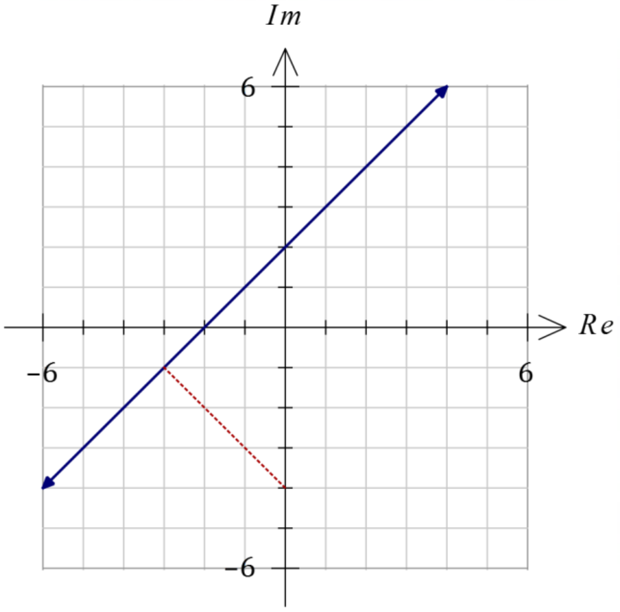


Solution
See graph
Specific behaviours
✓ endpoints and cusp at $x = 0$ ✓ symmetrical curves

Question 13 **(8 marks)**

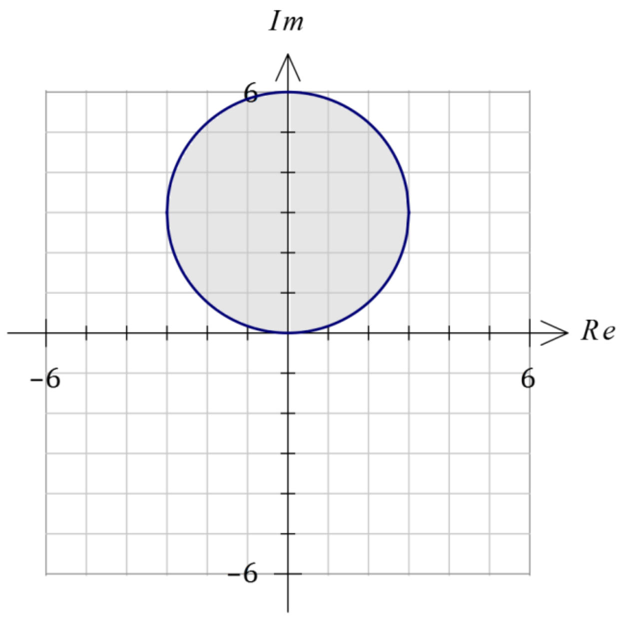
(a) On the Argand planes below sketch the locus of the complex number $z = x + iy$ given by

(i) $|z + 3 - 4i| = |z - 2 + i|$. (3 marks)



Solution
$|z - (-3 + 4i)| = |z - (2 - i)|$ See diagram.
Specific behaviours
✓ plots both points ✓ sketches perpendicular bisector ✓ correct axis intercepts

(ii) $|\bar{z} + 3i| \le 3$. (3 marks)



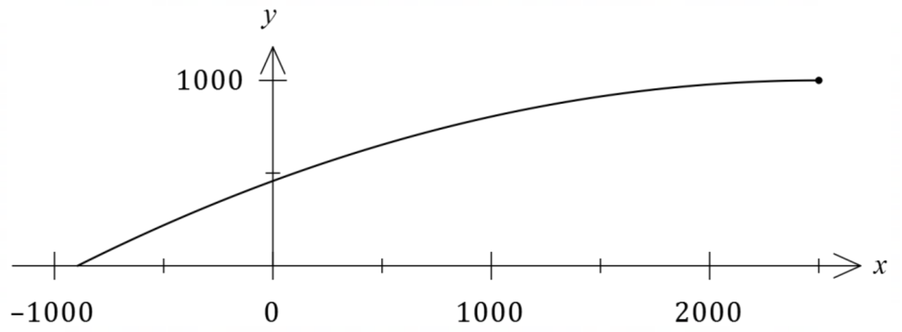
Solution
$|x - (y - 3)i| \le 3$ $x^2 + (y - 3)^2 \le 3^2$ See diagram.
Specific behaviours
✓ deals with conjugate ✓ indicates a shaded circle ✓ correct centre and radius

(b) For the locus $|z + 3 - 4i| = |z - 2 + i|$ in part (a), determine the minimum value for $|z + 4i|$. (2 marks)

Solution
Shortest distance from $z = -4i$ (on Im axis) to line. Hence minimum is $\sqrt{3^2 + 3^2} = 3\sqrt{2}$.
Specific behaviours
✓ indicates perpendicular distance to line ✓ correct minimum value

Question 14 **(10 marks)**

An aeroplane flying at a constant altitude releases a bomb at $2500\mathbf{i} + 1000\mathbf{j}$ with an initial velocity of $-240\mathbf{i}$. The path of the bomb is shown below.



Assume there is no wind in the region, air resistance can be ignored and the only acceleration acting on the bomb is $-10\mathbf{j}$ ms^{-2} due to gravity.

(a) Use the acceleration vector of the bomb to clearly deduce that its position vector at time t seconds after release is $\mathbf{r}(t) = (2500 - 240t)\mathbf{i} + (1000 - 5t^2)\mathbf{j}$. (3 marks)

Solution
$\mathbf{v}(t) = \int \begin{pmatrix} 0 \\ -10 \end{pmatrix} dt = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \mathbf{c_1}$ $\mathbf{v}(0) = \begin{pmatrix} -240 \\ 0 \end{pmatrix} \Rightarrow \mathbf{c_1} = \begin{pmatrix} -240 \\ 0 \end{pmatrix}, \quad \therefore \mathbf{v}(t) = \begin{pmatrix} -240 \\ -10t \end{pmatrix}$ $\mathbf{r}(t) = \int \begin{pmatrix} -240 \\ -10t \end{pmatrix} dt = \begin{pmatrix} -240t \\ -5t^2 \end{pmatrix} + \mathbf{c_2}$ $\mathbf{r}(0) = \begin{pmatrix} 2500 \\ 1000 \end{pmatrix} \Rightarrow \mathbf{c_2} = \begin{pmatrix} 2500 \\ 1000 \end{pmatrix}, \quad \therefore \mathbf{r}(t) = \begin{pmatrix} 2500 - 240t \\ 1000 - 5t^2 \end{pmatrix}$
Specific behaviours
✓ correct integration of acceleration vector, with constant   ✓ clearly shows use of initial conditions to obtain velocity vector   ✓ repeats with velocity vector to obtain position vector

(b) Determine the speed of the bomb 6 seconds after it is released. (2 marks)

Solution
$\mathbf{v}(6) = \begin{pmatrix} -240 \\ -60 \end{pmatrix}$ $\therefore s = \sqrt{(-240)^2 + (-60)^2}$ $= 60\sqrt{17} \approx 247.4 \text{ m/s}$
Specific behaviours
✓ indicates velocity   ✓ calculates correct speed

Three seconds after the bomb is released, a projectile is launched from the origin with a speed of v_0 at an angle of elevation of $\theta°$ to intercept it at a height of 680 m.

The position vector of the projectile T seconds after its launch is

$$\mathbf{r}(T) = (v_0 \cos(\theta)\, T)\mathbf{i} + (v_0 \sin(\theta)\, T - 5T^2)\mathbf{j}.$$

(c) Determine the value of v_0 and the value of θ so that the projectile intercepts the bomb. (5 marks)

Solution
Bomb reaches 680 m when $1000 - 5t^2 = 680 \Rightarrow t = 8\text{ s}$ Horizontal position of bomb is $2500 - 240(8) = 580$ m. Projectile will travel for $T = 8 - 3 = 5$ seconds. Horizontal position of projectile $v_0 \cos(\theta)\,(5) = 580 \Rightarrow v_0 \cos(\theta) = 116$ Vertical position of projectile $v_0 \sin(\theta)\,(5) - 5(5^2) = 680 \Rightarrow v_0 \sin(\theta) = 161$ Hence $\dfrac{v_0 \sin(\theta)}{v_0 \cos(\theta)} = \dfrac{161}{116} \Rightarrow \tan(\theta) = \dfrac{161}{116} \Rightarrow \theta \approx 54.23° \approx 0.946^r$ And $v_0 = 116 \div \cos 54.23° = 198.4\text{ m/s}$
Specific behaviours
✓ calculates time of interception ✓ uses horizontal position of interception to form equation ✓ uses vertical position of interception to form equation ✓ solves for angle in degrees or radians ✓ solves for initial speed

Question 15 **(6 marks)**

Points P, Q and R lie in plane Π with position vectors $\begin{pmatrix}2\\1\\5\end{pmatrix}$, $\begin{pmatrix}-2\\-2\\5\end{pmatrix}$ and $\begin{pmatrix}0\\0\\4\end{pmatrix}$ respectively.

(a) Determine the vector equation for plane Π in the form $\mathbf{r}\cdot\mathbf{n}=k$. (3 marks)

Solution

$$\overrightarrow{RP}=\begin{pmatrix}2\\1\\5\end{pmatrix}-\begin{pmatrix}0\\0\\4\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix},\quad \overrightarrow{RQ}=\begin{pmatrix}-2\\-2\\5\end{pmatrix}-\begin{pmatrix}0\\0\\4\end{pmatrix}=\begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

$$\mathbf{n}=\begin{pmatrix}2\\1\\1\end{pmatrix}\times\begin{pmatrix}-2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\-4\\-2\end{pmatrix}$$

$$k=\begin{pmatrix}3\\-4\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\4\end{pmatrix}=-8$$

Hence equation of Π is $\mathbf{r}\cdot\begin{pmatrix}3\\-4\\-2\end{pmatrix}=-8$

Specific behaviours

✓ obtains two vectors in the plane
✓ obtains normal to plane
✓ obtains value of k and states equation of plane

The equation of line L is $\mathbf{r}=\begin{pmatrix}3\\5\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}$.

(b) Determine, if possible, where line L intersects with plane Π. If not possible, explain why not. (3 marks)

Solution

Substitute equation of line into equation of plane:

$$\begin{pmatrix}3+2\lambda\\5-\lambda\\-2+5\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\-2\end{pmatrix}=-8$$

Hence

$$9+6\lambda-20+4\lambda+4-10\lambda=-8$$
$$-7=-8$$

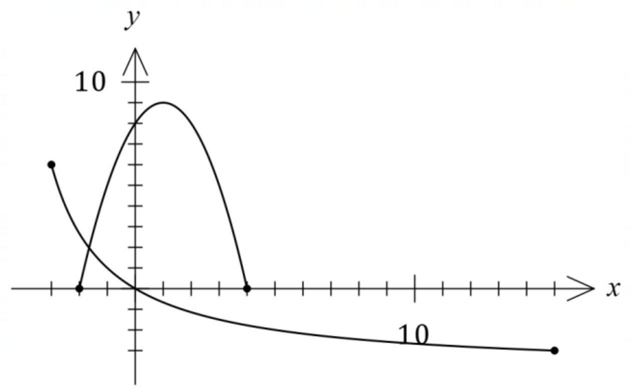
Since the equation is false but the solution of the equation is independent of λ then no values of λ will be a solution. Hence line L is parallel to plane Π but not in it and so there are no points of intersection.

Specific behaviours

✓ substitutes equation of line into equation of plane
✓ simplifies
✓ reasons that no points on the line lie in the plane

Question 16 **(8 marks)**

The graphs of $y = f(x)$ and $y = g(x)$ are shown at right.



The functions are defined by

$$f(x) = \frac{-4x}{x+5}, \qquad -3 \le x \le 15$$

and

$$g(x) = -x^2 + 2x + 8, \qquad -2 \le x \le 4.$$

(a) Explain why the inverse of g is not a function. (1 mark)

Solution
g is not a one-to-one function / g fails horizontal line test / etc.
Specific behaviours
✓ states valid reason

(b) Determine the definition for the inverse of f. (3 marks)

Solution
$x = \frac{-4y}{y+5}$ $xy + 5x + 4y = 0$ $y(x+4) = -5x$ $y = \frac{-5x}{x+4}, \quad -3 \le x \le 6.$
Specific behaviours
✓ interchanges x and y, cross multiplies and expands ✓ factors and obtains correct inverse ✓ limits domain to range of f

(c) Determine $g \circ f(-1)$. (1 mark)

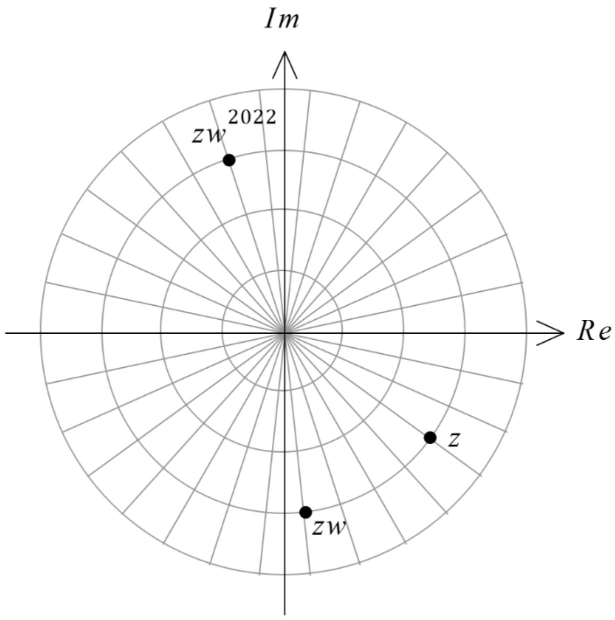
Solution
$g \circ f(-1) = g(1) = 9$
Specific behaviours
✓ correct value

(d) Determine the domain for the function $g \circ f(x)$. (3 marks)

Solution
$-2 \le R_f \le 4$ $\frac{-4x}{x+5} \ge -2 \Rightarrow x \le 5, \quad \frac{-4x}{x+5} \le 4 \Rightarrow x \ge -\frac{5}{2}$ $D_{g \circ f} = \left\{x \in \mathbb{R}, -\frac{5}{2} \le x \le 5\right\}$
Specific behaviours
✓ indicates restriction on range of f ✓ indicates one correct bound of range ✓ correct range

Question 17 **(5 marks)**

The complex number z is shown on the Argand diagram below and $w = \cos\left(-\frac{4\pi}{15}\right) + i\sin\left(-\frac{4\pi}{15}\right)$.



(a) Describe the geometric transformation performed by w when another complex number is multiplied by it, and plot and label zw on the Argand diagram. (2 marks)

Solution
w will rotate another complex number clockwise by $\frac{4\pi}{15}$ $(48°)$ about the origin (or rotate $-\frac{4\pi}{15}$ about the origin).
Specific behaviours
✓ correctly describes transformation ✓ correctly locates zw on diagram

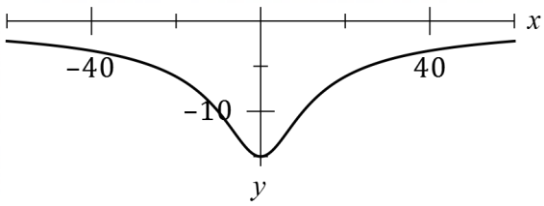
(b) Plot and label the complex number zw^{2022} on the Argand diagram. (3 marks)

Solution
$w^{2022} = \text{cis}\left(-\frac{8088\pi}{15}\right) = \text{cis}(-539.2\pi) = \text{cis}\big((540 - 539.2)\pi\big) = \text{cis}\left(\frac{12\pi}{15}\right)$ $\therefore \arg(zw^{2022}) = -\frac{3\pi}{15} + \frac{12\pi}{15} = \frac{9\pi}{15}$
Specific behaviours
✓ indicates correct argument of w^{2022} ✓ indicates correct argument of w^{2022} reduced to $-2\pi < \theta < 2\pi$ ✓ correctly locates zw^{2022} on diagram

Question 18 **(8 marks)**

The path of a small submersible moving below the surface of the sea (the x-axis) is shown in the diagram, where t is the time in seconds and $0 < t < 3\pi$.
The position vector of the submersible is

$$\mathbf{r}(t) = 9\cot\left(\frac{t}{3}\right)\mathbf{i} - 15\sin\left(\frac{t}{3}\right)\mathbf{j}\text{ m}.$$



(a) State, with reasoning, whether the submersible is moving from left to right or from right to left. (2 marks)

Solution
$\mathbf{v}(t) = -{}^{3}\!\Big/_{\sin^2\left(\frac{t}{3}\right)}\,\mathbf{i} - 5\cos\left(\frac{t}{3}\right)\mathbf{j}$ The i-coefficient will always be negative and so submersible is moving from right to left.
Specific behaviours
✓ differentiates to obtain velocity vector   ✓ states right to left, with reason

(b) Determine the Cartesian equation for the path of the submersible. (3 marks)

Solution
$\frac{\cos^2 A}{\sin^2 A} + \frac{\sin^2 A}{\sin^2 A} = \frac{1}{\sin^2 A} \Rightarrow \cot^2 A + 1 = \frac{1}{\sin^2 A}$ $x = 9\cot\left(\frac{t}{3}\right) \Rightarrow \cot\left(\frac{t}{3}\right) = \frac{x}{9}, \quad y = -15\sin\left(\frac{t}{3}\right) \Rightarrow \sin\left(\frac{t}{3}\right) = -\frac{y}{15}$ $\left(\frac{x}{9}\right)^2 + 1 = \left(-\frac{15}{y}\right)^2 \Leftrightarrow \left(\frac{x}{9}\right)^2 + 1 = \left(\frac{15}{y}\right)^2 \Leftrightarrow x^2y^2 + 81y^2 = 18\,225$
Specific behaviours
✓ obtains suitable trigonometric identity   ✓ sets $x = \cdots$, $y = \cdots$ and arranges equations for use with identity   ✓ eliminates trigonometric terms and simplifies

(c) Determine the distance travelled by the submersible when its depth below the surface is at least 7.5 metres, correct to the nearest centimetre. (3 marks)

Solution
Depth is at least 7.5 m when $15\sin\left(\frac{t}{3}\right) = 7.5 \Rightarrow \frac{\pi}{2} \le t \le \frac{5\pi}{2}$ Distance*: $d = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \sqrt{\left(-\frac{3}{\sin^2\left(\frac{t}{3}\right)}\right)^2 + \left(-5\cos\left(\frac{t}{3}\right)\right)^2}\,dt = 34.86\text{ m}$ * *Will take* $15\sim25$ *seconds to evaluate using numerical integration*
Specific behaviours
✓ obtains correct time interval   ✓ writes integral using magnitude of velocity   ✓ obtains correct distance, with units

Question 19 **(8 marks)**

The vector equation of sphere S_1 is $|\mathbf{r} - (-17\mathbf{i} + 15\mathbf{k})| = 22$. The position vector of the centre of sphere S_2 is $10\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ and the position vector of a point that lies on S_2 is $8\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$.

(a) Determine the Cartesian equation of sphere S_2. (2 marks)

Solution
$R = \|(10\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) - (8\mathbf{i} + 12\mathbf{j} + 6\mathbf{k})\|$ $= \sqrt{(2)^2 + (-6)^2 + (-9)^2}$ $= 11$ Cartesian equation: $(x - 10)^2 + (y - 6)^2 + (z + 3)^2 = 11^2$.
Specific behaviours
✓ calculates radius ✓ correct Cartesian equation

(b) The equation of line L_1 is $\mathbf{r} = -12\mathbf{i} + 24\mathbf{j} + 9\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Show that L_1 is tangential to S_1 and determine the position vector for the point of tangency. (4 marks)

Solution
$\|-12\mathbf{i} + 24\mathbf{j} + 9\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (-17\mathbf{i} + 15\mathbf{k})\| = 22$ $(3\lambda + 5)^2 + (2\lambda + 24)^2 + (2\lambda - 6)^2 = 22^2$ $17\lambda^2 - 102\lambda + 153 = 0$ $\lambda = -3$ As there is only one point of intersection then the line must be tangential to the sphere. The point of tangency is located at $\mathbf{r} = -12\mathbf{i} + 24\mathbf{j} + 9\mathbf{k} - 3(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ $= -21\mathbf{i} + 18\mathbf{j} + 3\mathbf{k}$
Specific behaviours
✓ substitutes equation of line into equation of sphere ✓ forms equation using magnitude ✓ solves for lambda and concludes tangential ✓ states point of tangency

(c) Show that S_1 and S_2 are tangential. (2 marks)

Solution
If spheres are tangential then the distance between their centres, d, will equal the sum of their radii $r_1 + r_2 = 22 + 11 = 33$: $d = \|(-17\mathbf{i} + 15\mathbf{k}) - (10\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})\|$ $= \sqrt{(-27)^2 + (6)^2 + (18)^2}$ $= 33$ Hence spheres are tangential.
Specific behaviours
✓ indicates expression for distance between centres and evaluates ✓ explains why spheres are tangential

End of questions

Supplementary page

Question number: _________

Supplementary page

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